Inequality

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Let a, b, c be positive real numbers such that

$a^2+b^2+c^2=9$, prove that $2(a+b+c)-abc \leq 10$.

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Note that number 10 isn't attainable upper bound for $2(a+b+c)-abc$ because for any positive a, b, c

such that $a^2+b^2+c^2=9$ holds inequality $2(a+b+c)-abc \leq 6\sqrt{2}$.

To prove that we will find maximal value of expression

$E := \dfrac{2(a+b+c)(a^2+b^2+c^2)-9abc}{\sqrt{(a^2+b^2+c^2)^3}}$.

Using normalization by $a+b+c=1$ and denoting $p=ab+bc+ca, q:=abc$ we obtain

$E=\dfrac{2(1-2p)-9q}{\sqrt{(1-2p)^3}}$. Note that $p=ab+bc+ca \leq \dfrac{(a+b+c)^2}{3}=\dfrac{1}{3}$

Since* $q \geq \max\left\{0, \dfrac{4p-1}{9}\right\}$ then $E \leq \dfrac{2(1-2p)-\max\{0,4p-1\}}{\sqrt{(1-2p)^3}}$.

For $p \in (0, 1/4]$ we obtain $E \leq \dfrac{2(1-2p)}{\sqrt{(1-2p)^3}}=\dfrac{2}{\sqrt{1-2p}} \leq \dfrac{2}{\sqrt{1-2\cdot 1/4}}=2\sqrt{2}$.

For $p \in [1/4, 1/3]$ we obtain $E \leq \dfrac{2(1-2p)-4p+1}{\sqrt{(1-2p)^3}}=\dfrac{3-8p}{\sqrt{(1-2p)^3}}$ and, we will prove that

$\dfrac{3-8p}{\sqrt{(1-2p)^3}} \leq 2\sqrt{2}$ as well. Indeed, $8(1-2p)^3-(3-8p)^2=(4p-1)(1+4p(1-4p)) \geq 0$.

Thus, $\max E=2\sqrt{2}$,that is for any positive a, b, c holds inequality

$$2(a+b+c)(a^2+b^2+c^2)-9abc \leq 2\sqrt{2}\sqrt{(a^2+b^2+c^2)^3}$$

and, using normalization by $a^2+b^2+c^2=9$ we obtain

$2(a+b+c)\cdot 9-9abc \leq 2\sqrt{2}\sqrt{9^3}=54\sqrt{2} \iff$

(1) $\quad 2(a+b+c)-abc \leq 6\sqrt{2}$.

Equality in inequality occurs iff one of three numbers equal 0 and two others equal $\dfrac{3}{\sqrt{2}}$.

* Inequality $9q \geq 4p-1$ is Schure Inequality $\sum a(a-b)(a-c) \geq 0$ in p,q-notation and normalized by $a+b+c=1$.